

FROM KIRCHBERG'S INEQUALITY TO THE GOLDBERG CONJECTURE

ANDREI MOROIANU

ABSTRACT. The main result of this note is that a compact Kähler manifold whose Ricci tensor has two distinct constant non-negative eigenvalues is locally the product of two Kähler-Einstein manifolds.

The problem of existence of Kähler metrics whose Ricci tensor has two distinct constant eigenvalues is related to the limiting case of Kirchberg's inequality [15] for the first eigenvalue of the Dirac operator on compact Kähler manifolds, as well as to the celebrated (still open) conjecture of Goldberg [13].

2000 *Mathematics Subject Classification.* Primary 53B20, 53C25

1. INTRODUCTION

In this note we consider compact Kähler manifolds (M, g, J) such that the symmetric endomorphism corresponding to the Ricci tensor Ric via the metric has two distinct constant eigenvalues. Since for Kähler manifolds the Ricci tensor is invariant under the action of J , each eigenvalue is of even multiplicity.

Such manifolds first appeared, to our knowledge, in a conjecture of Lichnerowicz concerning compact Kähler spin manifolds with least possible (compared to the scalar curvature) eigenvalue of the Dirac operator. These manifolds carry special spinor fields and it was showed in [17] that their Ricci tensor has two constant non-negative eigenvalues. In Section 2 we sketch the proof of the fact that their Ricci tensor has to be parallel. Then in Section 3 we analyze manifolds with constant eigenvalues of the Ricci tensor using tools coming from almost Kähler geometry, and obtain the following splitting theorem.

Theorem 1.1. [3] *Let (M, g, J) be a compact Kähler manifold whose Ricci tensor has exactly two distinct constant non-negative eigenvalues λ and μ . Then the universal cover of (M, g, J) is the product of two simply connected Kähler-Einstein manifolds of scalar curvatures λ and μ , respectively.*

Note that *local* irreducible examples of Kähler manifolds with eigenvalues of the Ricci tensor equal to 0 and 1 are known to exist in complex dimension two, cf. [8] and [2, Remark 1(c)]. Note also that the above result fails if one allows the Ricci tensor to have more than two different eigenvalues, as shown by the compact homogeneous Kähler manifolds (the generalized complex flag manifolds).

Another reason for this study came from an a priori unexpected link with the Goldberg conjecture [13], which states that any compact *Einstein* almost Kähler manifold is, in fact, Kähler-Einstein. For any Kähler manifold (M, g, J) with Ricci tensor having two distinct constant eigenvalues, one can define another g -orthogonal

The material of the present talk is based on a joint work with Vestislav Apostolov and Tedi Draghici.

almost complex structure \bar{J} , by changing the sign of J on one of the eigenspaces of Ric . The new almost complex structure \bar{J} , which is not integrable in general, commutes with J and has a closed fundamental 2-form, i.e. (g, \bar{J}) gives rise to an almost Kähler structure on the manifold. The integrability of \bar{J} holds precisely when the Ricci tensor of g is parallel, or equivalently, when g is locally a product of two Kähler–Einstein metrics; see Lemma 3.1. Moreover, any Kähler structure (g, J) with Ricci tensor having two distinct constant eigenvalues either both positive, or both negative, determines (and is determined by) a certain Einstein almost Kähler structure (\tilde{g}, \bar{J}) , see Corollary 3.3. In other words, any example of compact Kähler manifold whose Ricci tensor is non-parallel and has two constant eigenvalues of the same sign, would provide a counterexample to the Goldberg conjecture.

2. LIMITING MANIFOLDS FOR KIRCHBERG’S INEQUALITY

2.1. The Ricci tensor of limiting manifolds. We follow here the presentation and notations from [19].

Let (M^{2m}, g, J) be a compact Kähler spin manifold with positive scalar curvature s and suppose that its complex dimension m is even : $m = 2l$. K.-D. Kirchberg [15] showed that every eigenvalue λ of the Dirac operator on M satisfies

$$(1) \quad \lambda^2 \geq \frac{m}{4(m-1)} \inf_M s.$$

The manifolds which satisfy the limiting case of this inequality are called *limiting manifolds* for the remainder of this paper. In complex dimension 2, limiting manifolds were classified in 1994 by Th. Friedrich [11]. In all higher dimensions, they are characterized by the following

Theorem 2.1. (cf. [15]) *M is a limiting manifold if and only if its scalar curvature s is a positive constant and there exists a spinor $\Psi \in \Gamma(\Sigma^{l+1}M)$ such that*

$$(2) \quad \nabla_X \Psi = -\frac{1}{n}(X - iJX) \cdot D\Psi, \quad \forall X,$$

$$(3) \quad \nabla_X D\Psi = -\frac{1}{4}(\text{Ric}(X) + iJ\text{Ric}(X)) \cdot \Psi, \quad \forall X,$$

$$(4) \quad \kappa(X - iJX) \cdot \Psi = (\text{Ric}(X) - iJ\text{Ric}(X)) \cdot \Psi, \quad \forall X,$$

$$(5) \quad \kappa(X - iJX) \cdot D\Psi = (\text{Ric}(X) - iJ\text{Ric}(X)) \cdot D\Psi, \quad \forall X,$$

where $\kappa = \frac{s}{n-2}$. In particular, (2) implies (after a Clifford contraction) that $D\Psi \in \Gamma(\Sigma^l M)$.

These relations correspond to formulas (58), (59), (60) and (74) from [15], with the remark that Ψ above and ψ^{l-1} of [15] are related by $\Psi = j\psi^{l-1}$. (We recall, for the convenience of the reader, that Σ^l denotes the eigenspace for the Clifford action of the Kähler form on the spin bundle ΣM corresponding to the eigenvalue $i(m-2l)$).

In [17] we obtained the following

Theorem 2.2. (cf. [17], Thm.3.1) *The Ricci tensor of a limiting manifold of even complex dimension has two eigenvalues, κ and 0, the first one with multiplicity $n-2$ and the second one with multiplicity 2.*

Corollary 2.3. *The tangent bundle of M splits into a J -invariant orthogonal direct sum $TM = \mathcal{E} \oplus \mathcal{F}$ (where \mathcal{E} and \mathcal{F} are the eigenbundles of TM corresponding to the eigenvalues 0 and κ of Ric respectively). Moreover, the distributions \mathcal{E} and \mathcal{F} are integrable.*

Proof. All but the last statement are clear from Theorem 2.2, so we only prove the integrability of \mathcal{E} and \mathcal{F} . Let ρ denote the Ricci form of M , defined by $\rho(X, Y) = \text{Ric}(JX, Y)$, which, of course satisfies $d\rho = 0$. Remark that $X \lrcorner \rho = 0$ for $X \in \mathcal{E}$ and $X \lrcorner \rho = -\kappa JX$ for $X \in \mathcal{F}$. We consider arbitrary vector fields $X, Y \in \mathcal{E}$ and $Z \in \mathcal{F}$ and obtain (σ stands for the cyclic sum)

$$\begin{aligned} 0 &= d\rho(X, Y, Z) = \sigma(X(\rho(Y, Z)) - \rho([X, Y], Z)) \\ &= -\rho([X, Y], Z), \end{aligned}$$

so \mathcal{E} is integrable. Similarly, for $X, Y \in \mathcal{F}$ and $Z \in \mathcal{E}$ we have

$$\begin{aligned} 0 &= d\rho(X, Y, Z) = \sigma(X(\rho(Y, Z)) - \rho([X, Y], Z)) \\ &= Z(\rho(X, Y)) - \rho([Y, Z], X) - \rho([Z, X], Y) \\ &= \kappa(Z(g(JX, Y)) - g(J[Y, Z], X) - g(J[Z, X], Y)) \\ &= \kappa(g(\nabla_Y Z, JX) - g(\nabla_X Z, JY)) \\ &= -\kappa g([X, Y], JZ), \end{aligned}$$

which proves the integrability of \mathcal{F} . \square

From (2)–(5) follows that for every section X of \mathcal{E} we have

$$(6) \quad (X - iJX) \cdot \Psi = (X - iJX) \cdot D\Psi = 0$$

$$(7) \quad \nabla_X \Psi = \nabla_X D\Psi = 0.$$

2.2. Kählerian Killing spinors on Hodge manifolds. The theorem below is crucial for proving that the Ricci tensor of limiting manifolds is parallel. Its proof is based on the theory of projectable spinors and the classification of Spin^c manifolds with parallel spinors [18].

Definition 2.4. *Let (N^{4l-2}, g, J) be a Kähler manifold equipped with a Spin^c structure. A Kählerian Killing spinor on N is a spinor Ψ satisfying*

$$(8) \quad \nabla_X^A \Psi + \alpha X \cdot \Psi + i\alpha(-1)^l JX \cdot \bar{\Psi} = 0, \quad \forall X,$$

for some real constant $\alpha \neq 0$.

Theorem 2.5. *Let (N^n, g, J) , $n = 4l - 2$ be a simply connected compact Hodge manifold endowed with a Spin^c structure carrying a Kählerian Killing spinor $\Psi \in \Gamma(\Sigma^{l-1}N \oplus \Sigma^l N)$. Then this Spin^c structure on N is actually a spin structure.*

Proof. The proof is in two steps. We first show that the Kählerian Killing spinor on N induces a Killing spinor on some S^1 bundle over N , and then use the classification of Spin^c manifolds carrying Killing spinors to conclude.

The Hodge condition just means that $\frac{r}{2\pi}[\Omega] \in H^2(N, \mathbb{Z})$ for some $r \in \mathbb{R}^*$, and we will fix some r with this property. The isomorphism $H^2(N, \mathbb{Z}) \simeq H^1(N, S^1)$ guarantees the existence of some principal $U(1)$ bundle $\pi : S \rightarrow N$ whose first Chern class satisfies $c_1(S) = \frac{r}{2\pi}[\Omega]$. Furthermore, the Thom–Gysin exact sequence shows that S is simply connected if r is chosen in such a way that $\frac{r}{2\pi}[\Omega]$ is not a multiple of some integral class in $H^2(N, \mathbb{Z})$ (cf. [5], p.85).

The condition above on the first Chern class of S shows that there is a connection on S whose curvature form G satisfies $G = -ir\pi^*\Omega$. This connection induces a 1-parameter family of metrics on S which turn the bundle projection $\pi : S \rightarrow N$ into a Riemannian submersion with totally geodesic fibers. These metrics are given by

$$g_S^t(X, Y) = g(\pi_*(X), \pi_*(Y)) - t^2 \omega(X) \omega(Y) \quad (t > 0),$$

where ω denotes the (imaginary valued) connection form on S .

By pull-back from N , we obtain a Spin^c structure on S , whose spinor bundle is just $\pi^*\Sigma N$. After a straightforward computation, one finds that for a suitably chosen parameter t , the pull back $\pi^*\Psi$ of our Kählerian Killing spinor Ψ is a Killing spinor of the pull-back Spin^c structure on S .

Now, a standard argument shows that $\pi^*\Psi$ induces a parallel spinor Φ on the cone \bar{S} over S , endowed with the pull-back Spin^c structure (see [18]). Since S is compact, a theorem of Gallot ([12], Prop.3.1) shows that \bar{S} is an irreducible Riemannian manifold. From ([18], Thm.3.1) we then deduce that either the Spin^c structure of \bar{S} is actually a spin structure, or there exists a Kähler structure I on \bar{S} such that

$$(9) \quad X \cdot \Phi = iI(X) \cdot \Phi, \quad \forall X \in T\bar{S},$$

and the Spin^c structure of \bar{S} is the canonical Spin^c structure induced by I (these two cases do not exclude each other). In the first case we are done since the pull-back operation for Spin^c structures on Riemannian submersions is one-to-one. The second case turns out to be impossible, cf. [19] for the details. \square

We are now ready to complete the proof of:

Theorem 2.6. *The Ricci tensor of a limiting manifold of even complex dimension is parallel.*

Proof. We give here the main ideas, the reader is referred to [19] for details. Let N be a maximal leaf of the integrable distribution \mathcal{F} . An easy calculation shows that the Ricci tensor of N is defined positive. As N is complete, Myers' Theorem implies that N is compact and the theorem of Kobayashi ([16], Thm. A) shows that N is simply connected. We shall now consider the restriction Φ^N of $\Phi := \Psi + \frac{2}{\sqrt{n\kappa}} D\Psi$ to N . The relations (4) and (5) show that Φ^N is a section of the Spin^c structure on N with associated line bundle $\mathcal{E}^{1,0}|_N$.

A rather intricate computation yields that Φ^N is a Kählerian Killing spinor satisfying the hypothesis of Theorem 2.5. Moreover, N is a Hodge manifold: if we denote by i the inclusion $N \rightarrow M$ and by ρ the Ricci form of M , then $\kappa \Omega^N = i^*\rho$, which implies $\kappa[\Omega^N] = i^*(2\pi c_1(M))$, and thus $[\Omega^N]$ is a real multiple of $i^*(c_1(M)) \in H^2(N, \mathbb{Z})$.

We then apply Theorem 2.5 and deduce that the Spin^c structure on N has actually to be a *spin* structure (i.e. $\mathcal{E}^{1,0}|_N$ is a flat bundle on N). An explicit computation of the curvature of $\mathcal{E}^{1,0}|_N$ in terms of the curvature tensor of M yields that $TM|_N = \mathcal{E}|_N \oplus \mathcal{F}|_N$ is a parallel decomposition of the vector bundle $TM|_N$ over N , so finally, by choosing N arbitrarily, we deduce that E and F are parallel distributions on M . \square

3. ALMOST KÄHLER MANIFOLDS

We now relax the hypothesis in Theorem 2.6 : rather than assuming that M is a limiting manifold, we only suppose that the Ricci tensor of M has two constant non-negative eigenvalues.

3.1. The commuting almost Kähler structure. The main idea of the proof of Theorem 1.1 is to construct an almost Kähler structure \bar{J} on (M, g, J) , which is compatible with g and commutes with J .

Lemma 3.1. *Let (M, g, J) be a Kähler manifold whose Ricci tensor has constant eigenvalues $\lambda < \mu$. Denote by E_λ and E_μ the corresponding J -invariant eigenspaces and define a g -orthogonal almost complex structure \bar{J} by setting $\bar{J}|_{E_\lambda} = J|_{E_\lambda}$; $\bar{J}|_{E_\mu} = -J|_{E_\mu}$. Then J and \bar{J} mutually commute and (g, \bar{J}) is an almost Kähler structure, i.e. the fundamental form $\bar{\Omega}(\cdot, \cdot) = g(\bar{J}\cdot, \cdot)$ is symplectic. Moreover, (g, \bar{J}) is Kähler (i.e. \bar{J} is integrable) if and only if (M, g) is locally product of two Kähler-Einstein manifolds of scalar curvatures λ and μ , respectively.*

Proof. Denote by $\Omega(\cdot, \cdot) = g(J\cdot, \cdot)$ the fundamental form of (g, J) and consider the $(1,1)$ -forms α and β defined by

$$\alpha(X, Y) = \Omega(\text{pr}^\lambda(X), \text{pr}^\lambda(Y)), \quad \forall X, Y \in TM;$$

$$\beta = \Omega - \alpha,$$

where pr^λ (resp. pr^μ) denotes the orthogonal projection of the tangent bundle TM onto E_λ (resp. E_μ). The Ricci form $\rho(\cdot, \cdot) = \text{Ric}(J\cdot, \cdot)$ of (M, g, J) is then given by

$$\rho = \lambda\alpha + \mu\beta.$$

As $\Omega = \alpha + \beta$ and ρ are both closed $(1,1)$ -forms, so are the 2-forms α and β . By the very definition of \bar{J} , the fundamental form $\bar{\Omega}(\cdot, \cdot) = g(\bar{J}\cdot, \cdot)$ is given by

$$\bar{\Omega} = \alpha - \beta,$$

and hence is closed, i.e. $(g, \bar{J}, \bar{\Omega})$ is an almost Kähler structure; it is Kähler as soon as the Ricci tensor is parallel (equivalently, α and β are parallel), i.e. when (M, g) is locally a product of two Kähler-Einstein manifolds with scalar curvatures λ and μ , respectively. \square

Let us consider for a moment the more general context of Kähler manifolds (M, g, J) which admit a *commuting* almost Kähler structure \bar{J} . Any g -orthogonal almost complex structure \bar{J} which commutes with and differs from $\pm J$ gives rise to a g -orthogonal, J -invariant endomorphism $Q = -J \circ \bar{J}$ of TM such that $Q^2 = \text{Id}|_{TM}$; we thus define an orthogonal, J -invariant splitting of the tangent bundle TM

$$TM = E_+ \oplus E_-$$

into the sum of the ± 1 -eigenspaces of Q , the (complex) sub-bundles E_\pm , respectively. As in the proof of Lemma 3.1, we consider the $(1,1)$ -forms α and β , the restrictions of the fundamental form Ω of (g, J) to the spaces E_+ and E_- , respectively. The fundamental forms Ω and $\bar{\Omega}$ of (g, J) and (g, \bar{J}) are then given by

$$\Omega = \alpha + \beta; \quad \bar{\Omega} = \alpha - \beta,$$

proving that α and β are *closed*. Therefore, corresponding to any Kähler metric (g, J) admitting a commuting almost Kähler structure \bar{J} , we may consider a natural 1-parameter family g^t of metrics having the same property (see [19]):

$$(10) \quad g^t = g|_{E_+} + tg|_{E_-}, \quad t > 0,$$

where $g|_{E_+}$ (resp. $g|_{E_-}$) denotes the restriction of g to the eigenspaces E_+ (resp. to E_-).

Lemma 3.2. *For any $t > 0$, the metric g^t is Kähler with respect to J , almost Kähler with respect to \bar{J} and has the same Ricci tensor as the metric $g = g^1$.*

Proof. The first statements follow from the fact that the fundamental form of (g^t, J) (resp. (g^t, \bar{J})) is closed as being equal to $\alpha + t\beta$ (resp. $\alpha - t\beta$), where α and β are constructed as above with respect to $g = g^1$. For the last claim, note that the volume form of the metric g^t is a constant multiple of the volume form of $g = g^1$, so, from the local expression in complex coordinates, the Ricci forms of the Kähler structures (g^t, J) and (g, J) coincide. \square

As for Kähler metrics with Ricci tensor having distinct constant eigenvalues, Lemma 3.2 shows that one can deform any such given metric to one whose Ricci tensor has constant eigenvalues equal to $-1, 0$ or $+1$. In particular, we get

Corollary 3.3. *On a complex manifold (M, J) there is a one-to-one correspondence between Kähler metrics with Ricci tensor of constant eigenvalues $\lambda < \mu$ with $\lambda\mu > 0$ and Kähler-Einstein metrics \tilde{g} of scalar curvature 4λ carrying an orthogonal almost Kähler structure \bar{J} which commutes with and differs from $\pm J$; in this correspondence \bar{J} is compatible also with g and coincides (up to sign) with the almost Kähler structure defined in Lemma 3.1; moreover, \bar{J} is integrable precisely when g (and \tilde{g}) is locally product of two Kähler-Einstein metrics.*

Proof. Let (M, g, J) be a Kähler manifold whose Ricci tensor has constant eigenvalues $\lambda < \mu$ and \bar{J} be the almost Kähler structure commuting with J given by Lemma 3.1. It is easy to see that the ± 1 -eigenspaces of the endomorphism $Q = -J \circ \bar{J}$ above are given by $E_+ = E_\lambda$, $E_- = E_\mu$, where, we recall, E_λ and E_μ are the eigenspaces of Ric. By Lemma 2, the metric $\tilde{g} = g^{\mu/\lambda}$ obtained via (10) is Kähler-Einstein with scalar curvature 4λ . Conversely, starting from a Kähler-Einstein structure (\tilde{g}, J) of scalar curvature 4λ , endowed with an almost Kähler structure \bar{J} commuting with J , the deformation (10) provides a Kähler metric (g, J) whose Ricci tensor has constant eigenvalues $\lambda < \mu$, by putting $g^1 = \tilde{g}$ and $g = g^{\lambda/\mu}$. The almost complex structure \bar{J} is compatible to both g and \tilde{g} . It is clear then that the common Ricci tensor of g and \tilde{g} is \bar{J} -invariant, and therefore, \bar{J} coincides (up to sign) with the almost complex structure defined in Lemma 3.1. By Lemma 3.1 we also conclude that the integrability of \bar{J} is equivalent to g (hence also \tilde{g}) being locally a product of two Kähler-Einstein metrics. \square

3.2. Curvature obstructions to existence of strictly almost Kähler structures. The proof of Theorem 1.1 will be derived by showing the integrability of the almost Kähler structure obtained in Corollary 1. To do this we first observe that existence of a strictly almost Kähler structure imposes several non-trivial relations between different $U(n)$ -components of the curvature. Because the almost Kähler structure will take the center stage in what follows, we drop the bar-notation

from the previous sub-section and will even forget for now that in our situation the manifold also admits a Kähler structure.

Thus, let (M, g, J) be an almost Kähler manifold of (real) dimension $2n$. We start by reviewing some necessary elements of almost Kähler geometry.

The almost complex structure J gives rise to a type decomposition of complex vectors and forms, and accordingly, of any complex tensor field; by convention, J acts on the cotangent bundle T^*M by $Ja(X) = a(-JX)$. We thus have a decomposition of the complexified cotangent bundle

$$T^*M \otimes \mathbb{C} = \Lambda^{1,0}M \oplus \Lambda^{0,1}M,$$

and of the bundle of complex 2-forms

$$\Lambda^2M \otimes \mathbb{C} = \Lambda^{1,1}M \oplus \Lambda^{2,0}M \oplus \Lambda^{0,2}M.$$

A similar decomposition holds for the complex bundle $S^2M \otimes \mathbb{C}$ of symmetric 2-tensors. When considering *real* sections of Λ^2M (resp. of S^2M), we prefer to introduce the super-scripts ' and ' for denoting the projections to the real sub-bundles $\Lambda_{\mathbb{R}}^{1,1}M$ (resp. $S_{\mathbb{R}}^{1,1}M$) of J -invariant 2-forms (resp. symmetric 2-tensors) and to $[\Lambda^{2,0}M]$ (resp. $[S^{2,0}M]$) of J -anti-invariant ones (here and henceforth $[\]$ stands for the real vector bundle underlying a given complex bundle). Thus, for any section ψ of Λ^2M (resp. of S^2M) we have the splitting $\psi = \psi' + \psi''$, where

$$\psi'(\cdot, \cdot) = \frac{1}{2}(\psi(\cdot, \cdot) + \psi(J\cdot, J\cdot)) \quad \text{and} \quad \psi''(\cdot, \cdot) = \frac{1}{2}(\psi(\cdot, \cdot) - \psi(J\cdot, J\cdot)).$$

Note that $\Lambda_{\mathbb{R}}^{1,1}M$ can be identified with $S_{\mathbb{R}}^{1,1}M$ via the complex structure J : for any $\alpha \in \Lambda_{\mathbb{R}}^{1,1}M$,

$$A = (J \circ \alpha) := \alpha(J\cdot, \cdot)$$

is the corresponding section of $S_{\mathbb{R}}^{1,1}M$.

The real bundle $[\Lambda^{2,0}M]$ (resp. $[S^{2,0}M]$) inherits a canonical complex structure J , acting by

$$(J\psi)(X, Y) := -\psi(JX, Y), \quad \forall \psi \in [\Lambda^{2,0}M].$$

(We adopt a similar definition for the action of J on $[S^{2,0}M]$).

It is well known that the fundamental form $\Omega(\cdot, \cdot) = g(J\cdot, \cdot)$ of an almost Kähler structure is a real harmonic 2-form of type $(1, 1)$, i.e. satisfies:

$$\Omega(J\cdot, J\cdot) = \Omega(\cdot, \cdot), \quad d\Omega = 0 \quad \text{and} \quad \delta\Omega = 0,$$

where d and δ are the differential and co-differential operators acting on forms. Moreover, if ∇ is the Levi-Civita connection of g , then $\nabla\Omega$ (which is identified with the Nijenhuis tensor of J) is a section of the real vector bundle $[\Lambda^{1,0}M \otimes \Lambda^{2,0}M]$.

We first derive several consequences from the classical Weitzenböck formula for a 2-form ψ :

$$(11) \quad \begin{aligned} \Delta\psi - \nabla^*\nabla\psi &= [\text{Ric}(\Psi\cdot, \cdot) - \text{Ric}(\cdot, \Psi\cdot)] - 2R(\psi) \\ &= \frac{2(n-1)}{n(2n-1)}s\psi - 2W(\psi) + \frac{(n-2)}{(n-1)}[\text{Ric}_0(\Psi\cdot, \cdot) - \text{Ric}_0(\cdot, \Psi\cdot)], \end{aligned}$$

where: $\Delta = d\delta + \delta d$ denotes the Riemannian Laplace operator acting on 2-forms, ∇^* denotes the adjoint of ∇ with respect to g ; $\text{Ric}_0 = \text{Ric} - \frac{s}{2n}g$ is the traceless part of the Ricci tensor, $s = \text{trace}(\text{Ric})$ is the scalar curvature, Ψ is the skew-symmetric endomorphism of TM identified to ψ via the metric, and R and W are respectively

the curvature tensor and the Weyl tensor, considered as endomorphisms of $\Lambda^2 M$ or as sections of $\Lambda^2 M \otimes \Lambda^2 M$, depending on the context.

Applying relation (11) to the (harmonic) fundamental form Ω of the almost Kähler structure (g, J) , we obtain

$$(12) \quad \nabla^* \nabla \Omega = 2R(\Omega) - [\text{Ric}(J\cdot, \cdot) - \text{Ric}(\cdot, J\cdot)] .$$

Note that the Ricci tensor of a Kähler structure is J -invariant, but this is no longer true for an arbitrary almost Kähler structure. It will be thus useful to introduce the invariant and the anti-invariant parts of the Ricci tensor with respect to the almost complex structure J , Ric' and Ric'' , respectively. We also put

$$\rho = J \circ \text{Ric}'$$

to be the (1,1)-form corresponding to the J -invariant part of Ric , which will be called *Ricci form* of (M, g, J) . For Kähler manifolds, ρ is clearly equal to the image of Ω under the action of the curvature R , but this is not longer true for almost Kähler manifolds. In fact,

$$\rho^* = R(\Omega)$$

can be considered as a second (twisted) Ricci form of (M, g, J) which is not, in general, J -invariant (see e.g. [22]). We will consequently denote by $(\rho^*)'$ and $(\rho^*)''$ the corresponding 2-forms which are sections of the bundles $\Lambda_{\mathbb{R}}^{1,1} M$ and $\llbracket \Lambda^{2,0} M \rrbracket$, respectively. With these notations, formula (12) is a measure of the difference of the two types of Ricci forms on an almost Kähler manifold:

$$(13) \quad \rho^* - \rho = \frac{1}{2}(\nabla^* \nabla \Omega).$$

Taking the inner product with Ω of the relation (13) we obtain the difference of the two types of scalar curvatures:

$$(14) \quad s^* - s = |\nabla \Omega|^2 = \frac{1}{2} |\nabla J|^2 ,$$

where, we recall $s = \text{trace}(\text{Ric})$ is the usual scalar curvature of g , and $s^* = 2\langle R(\Omega), \Omega \rangle$ is the so-called *star-scalar curvature* of the almost Kähler structure (g, J) . Here and throughout the paper, the inner product induced by the metric g on various tensor bundles over the manifold will be denoted by $\langle \cdot \rangle$, while the corresponding norm is denoted by $|\cdot|$; note that $\langle \cdot \rangle$ acting on 2-forms differs by a factor of 1/2 compared to when it acts on corresponding tensors or endomorphisms. In the present paper $\nabla \Omega$ is viewed as a $\Lambda^2 M$ -valued 1-form, while ∇J is considered as a section of $(T^* M)^{\otimes 2} \otimes TM$, etc.

Formulae (13) and (14) can be interpreted as “obstructions” to the (local) existence of a strictly almost Kähler structure J , compatible with a given metric g ; see e.g. [4]. We derived these relations by using properties of the 2-jet of J (although eventually (14) depends on the 1-jet only), so that (13) and (14) can be viewed as obstructions to the lifting of the 0-jet of J to the 2-jet.

In fact, there is even a more general identity than (13), due to Gray [14], which could also be interpreted as an obstruction to the lifting of the 0-jet of J to the 2-jet: Starting from the splitting

$$\Lambda^2 M = \Lambda_{\mathbb{R}}^{1,1} M \oplus \llbracket \Lambda^{2,0} M \rrbracket ,$$

we denote by \tilde{R} the component of the curvature operator acting trivially on the first factor, i.e.

$$\tilde{R}_{X,Y,Z,T} = \frac{1}{4} \left(R_{X,Y,Z,T} - R_{JX,JY,Z,T} - R_{X,Y,JZ,JT} + R_{JX,JY,JZ,JT} \right).$$

Thus, \tilde{R} can be viewed as a section of the bundle $\text{End}_{\mathbb{R}}(\llbracket \Lambda^{2,0} M \rrbracket)$, which in turn decomposes further as

$$\text{End}_{\mathbb{R}}(\llbracket \Lambda^{2,0} M \rrbracket) = \left(\text{End}_{\mathbb{R}}(\llbracket \Lambda^{2,0} M \rrbracket) \right)' \oplus \left(\text{End}_{\mathbb{R}}(\llbracket \Lambda^{2,0} M \rrbracket) \right)'',$$

into the sub-bundles of endomorphisms of $\llbracket \Lambda^{2,0} M \rrbracket$ which commute, respectively, anti-commute with the action of J on $\llbracket \Lambda^{2,0} M \rrbracket$. Denoting by \tilde{R}' and \tilde{R}'' the corresponding components of \tilde{R} , Gray's identity is [14]

$$(15) \quad \tilde{R}' = -\frac{1}{4} \sum (\nabla_{e_j} \Omega) \otimes (\nabla_{e_j} \Omega).$$

As for the component \tilde{R}'' , from its definition we have

$$\begin{aligned} (\tilde{R}'')_{X,Y,Z,T} = \frac{1}{8} & \left(R_{X,Y,Z,T} - R_{JX,JY,Z,T} - R_{X,Y,JZ,JT} + R_{JX,JY,JZ,JT} \right. \\ & \left. + R_{X,JY,Z,JT} + R_{JX,Y,Z,JT} + R_{X,JY,JZ,T} + R_{JX,Y,JZ,T} \right), \end{aligned}$$

showing that

$$(\tilde{R}'')_{Z_1,Z_2,Z_3,Z_4} = R_{Z_1,Z_2,Z_3,Z_4} = W_{Z_1,Z_2,Z_3,Z_4} \quad \forall Z_i \in T^{1,0} M.$$

Thus, \tilde{R}'' is actually determined by the Weyl curvature of M .

The next Weitzenböck-type formula provides a further obstruction, this time to the lift of the 3-jet of J to the 4-jet. We skip the proof, which can be found in [3].

Proposition 3.4. *For any almost Kähler structure (g, J, Ω) the following relation holds:*

$$(16) \quad \begin{aligned} \Delta(s^* - s) = & -4\delta(J\delta(J\text{Ric}'')) + 8\delta(\langle \rho^*, \nabla \cdot \Omega \rangle) + 2|\text{Ric}''|^2 \\ & - 8|\tilde{R}''|^2 - |\nabla^* \nabla \Omega|^2 - |\phi|^2 + 4\langle \rho, \phi \rangle - 4\langle \rho, \nabla^* \nabla \Omega \rangle, \end{aligned}$$

where the semi-positive $(1,1)$ -form ϕ is given by $\phi(X, Y) = \langle \nabla_{JX} \Omega, \nabla_Y \Omega \rangle$; δ denotes the co-differential with respect to ∇ , acting on 1-forms and 2-tensors.

Integrating (16) over the manifold, one obtains an integral formula identical to the one in [21, Proposition 3.2], up to some integration by parts. In particular we have:

Corollary 3.5. ([21]) *For any compact almost Kähler manifold with J -invariant Ricci tensor the following inequality holds:*

$$(17) \quad \int_M [4\langle \rho, \phi \rangle - 4\langle \rho, \nabla^* \nabla \Omega \rangle - |\nabla^* \nabla \Omega|^2 - |\phi|^2] dV_g \geq 0,$$

where $dV_g = \frac{1}{n!} \Omega^n$ is the volume form of g .

Remark 3. As shown by Sekigawa, the above inequality gives an obstruction to the (global) existence of strictly almost Kähler structures, when the metric g is Einstein with non-negative scalar curvature. Indeed, in this case $\text{Ric}'' = 0$ and $2\langle \rho, \phi \rangle = \langle \rho, \nabla^* \nabla \Omega \rangle = \frac{s}{2n} |\nabla \Omega|^2$, so that, by (17), $\nabla \Omega = 0$, i.e. J is necessarily

Kähler. In dimension 4, other integrability results have been derived from (16); see e.g. [10], [1], [20].

Proof of Theorem 1.1. We now turn back to the notation used in Section 3.1. Thus, (g, J, Ω) denotes the Kähler structure, with Ricci tensor having two non-negative distinct constant eigenvalues $0 \leq \lambda < \mu$, while $(g, \bar{J}, \bar{\Omega})$ is the almost Kähler structure constructed by Lemma 3.1; we shall also use the (1,1)-forms α and β introduced in Section 2.1, so that we have

$$(18) \quad \Omega = \alpha + \beta; \quad \bar{\Omega} = \alpha - \beta; \quad \rho = \lambda\alpha + \mu\beta; \quad \bar{\rho} = \lambda\alpha - \mu\beta,$$

where ρ and $\bar{\rho}$ are the Ricci forms of (g, J) and (g, \bar{J}) , respectively.

For proving Theorem 1.1 it is enough to show that \bar{J} is integrable (see Lemma 3.1), or equivalently, that $\nabla\bar{\Omega} = 0$. The latter will be derived from the integral inequality stated in Corollary 3.5 (see Remark 3).

Let $\bar{\phi}(X, Y) = \langle \nabla_{\bar{J}X}\bar{\Omega}, \nabla_Y\bar{\Omega} \rangle$ be the semi-positive definite (1,1)-form with respect to \bar{J} , defined in Proposition 3.4. By (18) and using the semi-positivity of the (1,1)-forms α and $\bar{\phi}$, we get

$$\begin{aligned} (19) \quad \langle \bar{\rho}, \bar{\phi} \rangle - \langle \bar{\rho}, \nabla^*\nabla\bar{\Omega} \rangle &= (\lambda - \mu)\langle \alpha, \bar{\phi} \rangle + (\mu - \lambda)\langle \alpha, \nabla^*\nabla\bar{\Omega} \rangle \\ &\quad + \mu\langle \bar{\Omega}, \bar{\phi} \rangle - \mu\langle \bar{\Omega}, \nabla^*\nabla\bar{\Omega} \rangle \\ &= (\lambda - \mu)\langle \alpha, \bar{\phi} \rangle + (\mu - \lambda)\langle \alpha, \nabla^*\nabla\bar{\Omega} \rangle - \frac{\mu}{2}|\nabla\bar{\Omega}|^2 \\ &\leq (\mu - \lambda)\langle \alpha, \nabla^*\nabla\bar{\Omega} \rangle - \frac{\mu}{2}|\nabla\bar{\Omega}|^2. \end{aligned}$$

Since $\langle \alpha, \nabla\bar{\Omega} \rangle = 0$ (because α and $\nabla\bar{\Omega}$ are of type (1,1) and (2,0) + (0,2), respectively), we have

$$\langle \alpha, \nabla^*\nabla\bar{\Omega} \rangle = \langle \nabla\alpha, \nabla\bar{\Omega} \rangle = \frac{1}{2}|\nabla\bar{\Omega}|^2,$$

where in the last step we used that $\alpha = \frac{1}{2}(\Omega + \bar{\Omega})$ and Ω is parallel. Substituting into the inequality (19), we obtain

$$\langle \bar{\rho}, \bar{\phi} \rangle - \langle \bar{\rho}, \nabla^*\nabla\bar{\Omega} \rangle \leq -\frac{\lambda}{2}|\nabla\bar{\Omega}|^2.$$

Since by assumption $\lambda \geq 0$, the latter inequality shows that $\langle \bar{\rho}, \bar{\phi} \rangle - \langle \bar{\rho}, \nabla^*\nabla\bar{\Omega} \rangle$ is an everywhere non-positive function and Corollary 3.5 then implies that $\nabla^*\nabla\bar{\Omega} = 0$; after multiplying by $\bar{\Omega}$ we reach $\nabla\bar{\Omega} = 0$. \square

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ANDREI MOROIANU, CMAT, ÉCOLE POLYTECHNIQUE, UMR 7640 DU CNRS, 91128 PALAISEAU, FRANCE

E-mail address: am@math.polytechnique.fr